

MULTIPLE POSITIVE SOLUTIONS OF PARABOLIC SYSTEMS WITH NONLINEAR, NONLOCAL INITIAL CONDITIONS

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ABSTRACT. In this paper we study the existence, localization and multiplicity of positive solutions for parabolic systems with nonlocal initial conditions. In order to do this, we extend an abstract theory that was recently developed by the authors jointly with Radu Precup, related to the existence of fixed points of nonlinear operators satisfying some upper and lower bounds. Our main tool is the Granas fixed point index theory. We also provide a non-existence result and an example to illustrate our theory.

1. INTRODUCTION

In this paper we deal with the existence, non-existence and localization of positive solutions of the following system of parabolic equations subject to nonlinear, nonlocal initial conditions

$$(1.1) \quad \begin{cases} u_t - \Delta u = f(t, x, u, v), & (t, x) \in (0, t_{\max}) \times \Omega, \\ v_t - \Delta v = g(t, x, u, v), & (t, x) \in (0, t_{\max}) \times \Omega, \\ u(t, x) = v(t, x) = 0, & (t, x) \in (0, t_{\max}) \times \partial\Omega, \\ u(0, \cdot) = \alpha(u, v), \\ v(0, \cdot) = \beta(u, v), \end{cases}$$

where $\Omega \subset \mathbb{R}^m$ is a bounded domain that is Dirichlet regular, $f, g: (0, t_{\max}) \times \Omega \times \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are continuous functions and α and β belong to a fairly general class of nonlinear operators that covers a variety of cases, a *particular* example being

$$(1.2) \quad \alpha(u, v)(x) = \int_0^{t_{\max}} u(t, x) dt, \quad \beta(u, v)(x) = \int_0^{t_{\max}} v(t, x) dt.$$

Initial nonlocal conditions have been investigated in a variety of settings, for example in the case of multi-point [7], integral [6, 10, 19, 20, 22, 23] and nonlinear conditions [2, 3, 4, 5], see also the recent review [24]. In particular, a physical motivation for the integral form of the initial condition is given in [19] for the one-dimensional heat equation. Furthermore a number of applications of nonlocal problems for evolution equations are illustrated in Section 10.2 of [18].

In a recent paper [16] the authors investigated the existence, localization and multiplicity of positive solutions of systems of (p, q) -Laplacian equations subject to Dirichlet boundary conditions. The main tool in [16] is the development of a general abstract framework for the existence of fixed points of nonlinear operators acting on cones that

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satisfy an inequality of Harnack-type. Within this setting, the authors of [16] used the Granas fixed point index (see for example [9, 13]) and, in order to compute the index, they used some estimates from above, using the norm, and from below, utilizing a seminorm. Here we *extend* the theoretical results of [16] to a more general setting; this generalization is *apparently* simple but fruitful and is motivated by the application to the parabolic system (1.1). In particular, we replace the use of the seminorm with the use of a more general positively homogeneous functional and, moreover, we relax the assumptions on the cone. The Remarks 2.5, 3.2, 3.11 and 3.17 illustrate in details the differences between the two theoretical approaches and their applicability. We point out that our new approach is quite general and covers, *as a special case*, the system (1.1).

Inspired by the setting considered in [21], where the problem of *one* parabolic equation with nonlocal, *linear* initial condition was studied, we apply our theoretical results to the parabolic system (1.1). In contrast with the paper [16], where the space L^∞ with an integral seminorm was used, here, in order to seek mild solutions of our problem, we use the classical space of continuous functions, with a very natural positively homogeneous functional, namely the minimum on a suitable subset. A similar idea has been used with success in the context of ordinary differential equations and integral equations, see for example [14]. In our case, a key role for our multiplicity results is played by a weak Harnack-type inequality, see Remark 3.15.

In the case of the system (1.1) we obtain existence, localization, multiplicity and non-existence of positive mild solutions.

We illustrate in an example the applicability of our results and we show that the constants that occur in our theory can be computed.

2. ABSTRACT EXISTENCE THEOREMS

Let $(E_i, |\cdot|_i)$ ($i = 1, 2$) be Banach spaces and let $[\cdot]_i$ be positively homogeneous continuous functionals on E_i . In what follows, we omit the subscript in $[\cdot]_i$, when confusion is unlikely.

Let also $G_i \subset E_i$ be closed convex wedges, which is understood to mean that

$$\lambda u + \mu v \in G_i \text{ for all } u, v \in G_i \text{ and } \lambda, \mu \geq 0.$$

Moreover, let $K_i \subset G_i$ be closed convex cones, which means that K_i are closed convex wedges such that $K_i \cap (-K_i) = \{0\}$. The wedges induce the natural semiorders \preceq on E_i in the following way:

$$u \preceq v \text{ if and only if } v - u \in G_i, u, v \in E_i.$$

By a semiorder we mean that the relation \preceq is reflexive and transitive, but not necessarily antisymmetric.

We assume that the functionals $[\cdot]$ are monotone on K_i with respect to the semiorder \preceq , that is for $i \in \{1, 2\}$ we have

$$(2.1) \quad \text{if } u \preceq v \text{ then } [u] \leq [v] \text{ for } u, v \in K_i.$$

In particular we have $[u] \geq 0$ for $u \in K_i$.

We assume that there exist some elements $\psi_i \in K_i$ such that $|\psi_i| = 1$ and for $i \in \{1, 2\}$

$$(2.2) \quad u \preceq |u|\psi_i \text{ for all } u \in K_i.$$

Note that (2.1) and (2.2) yield

$$(2.3) \quad \lfloor u \rfloor_i \leq \lfloor \psi \rfloor |u| \text{ for all } u \in K_i.$$

In particular, we have $\lfloor \psi_i \rfloor > 0$ if $\lfloor \cdot \rfloor$ is nonzero.

In what follows by the *compactness* of a continuous operator we mean the relative compactness of its range. By the *complete continuity* of a continuous operator we mean the relative compactness of the image of every bounded set of the domain.

We seek the fixed points of a completely continuous operator

$$N := (N_1, N_2): K_1 \times K_2 \rightarrow K_1 \times K_2,$$

that is $(u, v) \in K_1 \times K_2$ such that $N(u, v) = (u, v)$.

We shall discuss not only the existence, but also the localization and multiplicity of the solutions of the nonlinear equation $N(u, v) = (u, v)$. In order to do this, we utilize the Granas fixed point index, $\text{ind}_C(f, U)$ (for more information on the index and its applications we refer the reader to [9, 13]).

The next Proposition describes some of the useful properties of the index, for details see Theorem 6.2, Chapter 12 of [13].

Proposition 2.1. *Let C be a closed convex subset of a Banach space, $U \subset C$ be open in C and $f: \bar{U} \rightarrow C$ be a compact map with no fixed points on the boundary ∂U of U . Then the fixed point index has the following properties:*

- (i) *(Existence)* If $\text{ind}_C(f, U) \neq 0$ then $\text{fix}(f) \neq \emptyset$, where $\text{fix} f = \{x \in \bar{U} : f(x) = x\}$.
- (ii) *(Additivity)* If $\text{fix} f \subset U_1 \cup U_2 \subset U$ with U_1, U_2 open in C and disjoint, then

$$\text{ind}_C(f, U) = \text{ind}_C(f, U_1) + \text{ind}_C(f, U_2).$$

- (iii) *(Homotopy invariance)* If $h: \bar{U} \times [0, 1] \rightarrow C$ is a compact homotopy such that $h(u, t) \neq u$ for $u \in \partial U$ and $t \in [0, 1]$ then

$$\text{ind}_C(h(\cdot, 0), U) = \text{ind}_C(h(\cdot, 1), U).$$

- (iv) *(Normalization)* If f is a constant map, with $f(u) = u_0$ for every $u \in \bar{U}$, then

$$\text{ind}_C(f, U) = \begin{cases} 1, & \text{if } u_0 \in U \\ 0, & \text{if } u_0 \notin \bar{U}. \end{cases}$$

In particular, $\text{ind}_C(f, C) = 1$ for every compact function $f: C \rightarrow C$, since f is homotopic to any $u_0 \in C$, by the convexity of C (take $h(u, t) = tf(u) + (1-t)u_0$).

2.1. Fixed point results. We begin with two theorems on the existence and localization of one solution of the operator equation $N(u, v) = (u, v)$. Set $K = K_1 \times K_2$ and

$$C = C(R_1, R_2) := \{(u, v) \in K_1 \times K_2 : |u| \leq R_1, |v| \leq R_2\},$$

for some fixed numbers R_1, R_2 .

The first Theorem is a generalization of Theorem 2.17 of [16].

Theorem 2.2. Assume that there exist numbers r_i, R_i , $i = 1, 2$ with $0 < r_i < \lfloor \psi_i \rfloor R_i$ such that

$$(2.4) \quad \inf_{\substack{(u,v) \in C \\ \lfloor u \rfloor = r_1, \lfloor v \rfloor \leq r_2}} \lfloor N_1(u, v) \rfloor > r_1, \quad \inf_{\substack{(u,v) \in C \\ \lfloor u \rfloor \leq r_1, \lfloor v \rfloor = r_2}} \lfloor N_2(u, v) \rfloor > r_2,$$

and

$$(2.5) \quad \sup_{(u,v) \in C} |N_i(u, v)| \leq R_i \quad (i = 1, 2).$$

Then N has at least one fixed point $(u, v) \in K_1 \times K_2$ such that $|u| \leq R_1$, $|v| \leq R_2$ and either $\lfloor u \rfloor > r_1$ or $\lfloor v \rfloor > r_2$.

Proof of Theorem 2.2. The assumption (2.5) implies that $N(C) \subset C$. Therefore, by Proposition 2.1, we obtain $\text{ind}_C(N, C) = 1$. Let

$$U := \{(u, v) \in C : \lfloor u \rfloor < r_1, \lfloor v \rfloor < r_2\}.$$

This is an open set, whose boundary ∂U with respect to C is equal to $\partial U = A_1 \cup A_2$, where

$$\begin{aligned} A_1 &= \{(u, v) \in C : \lfloor u \rfloor = r_1, \lfloor v \rfloor \leq r_2\}, \\ A_2 &= \{(u, v) \in C : \lfloor u \rfloor \leq r_1, \lfloor v \rfloor = r_2\}. \end{aligned}$$

Observe that (2.4) implies that there are no fixed points of N on ∂U . Therefore, the indices $\text{ind}_C(N, U)$ and $\text{ind}_C(N, C \setminus \overline{U})$ are well defined and their sum, by the additivity property of the index, is equal to one. Therefore, it suffices to prove that $\text{ind}_C(N, U) = 0$. Take $h = (R_1\psi_1, R_2\psi_2) \in C$ and consider the homotopy $H : C \times [0, 1] \rightarrow C$,

$$H(u, v, t) := th + (1 - t)N(u, v).$$

We claim that H is fixed point free on ∂U . Since

$$(2.6) \quad \lfloor R_i\psi_i \rfloor = R_i \lfloor \psi_i \rfloor > r_i, \quad i = 1, 2.$$

we have $(u, v) \neq h = H(u, v, 1)$ for all $(u, v) \in \partial U$. It remains to show that $H(u, v, t) \neq (u, v)$ for $(u, v) \in \partial U$ and $t \in (0, 1)$. Assume the contrary. Then there exists $(u, v) \in A_1 \cup A_2$ and $t \in (0, 1)$ such that

$$(2.7) \quad (u, v) = th + (1 - t)N(u, v).$$

Suppose that $(u, v) \in A_1$. Then,

$$N_1(u, v) \preceq |N_1(u, v)|\psi_1 \preceq R_1\psi_1$$

and exploiting the first coordinate of the equation (2.7), we obtain

$$(2.8) \quad u = N_1(u, v) + t(R_1\psi_1 - N_1(u, v)) \succeq N_1(u, v).$$

Using the monotonicity of $\lfloor \cdot \rfloor$ and (2.4) we obtain $r_1 = \lfloor u \rfloor \geq \lfloor N_1(u, v) \rfloor > r_1$, which is impossible. Similarly, we derive a contradiction if $(u, v) \in A_2$.

By the homotopy invariance of the index we obtain $\text{ind}_C(N, U) = \text{ind}_C(h, U)$. From (2.6) we have $h \notin \overline{U}$, hence $\text{ind}_C(N, U) = \text{ind}_C(h, U) = 0$, as we wished. □

Remark 2.3. From the proof we can deduce that if we change the assumption (2.4) into

$$(2.9) \quad \inf_{\substack{(u,v) \in C \\ \lfloor u \rfloor = r_1, \lfloor v \rfloor \leq r_2}} \lfloor N_1(u, v) \rfloor \geq r_1, \quad \inf_{\substack{(u,v) \in C \\ \lfloor u \rfloor \leq r_1, \lfloor v \rfloor = r_2}} \lfloor N_2(u, v) \rfloor \geq r_2,$$

then we obtain at least one fixed point $(u, v) \in C$ with slightly weaker localization: $\lfloor u \rfloor \geq r_1$ or $\lfloor v \rfloor \geq r_2$. The assumption (2.9) permits the existence of fixed points of N on ∂U . The assumption (2.4) is more convenient when dealing with multiplicity results.

Remark 2.4. We observe that, using the relation (2.3), a lower bound for the solution in terms of the functional $\lfloor \cdot \rfloor$ provides a lower bound for the norm of the solution, namely

$$\lfloor u \rfloor > r_1 \implies |u| > \lfloor \psi_1 \rfloor^{-1} r_1, \quad \lfloor v \rfloor > r_2 \implies |v| > \lfloor \psi_2 \rfloor^{-1} r_2.$$

Remark 2.5. The main differences between Theorem 2.2 and Theorem 2.17 of [16] consist in:

- The possibility of considering a positively homogeneous functional $\lfloor \cdot \rfloor$ instead of a seminorm.
- The assumption on the cone; in Theorem 2.17 of [16] it is needed the existence of ψ such that $u \leq |u|\psi_i$ for all $u \in E_i$, where \leq is the order induced by the cone K_i , $i = 1, 2$. Here, instead, we can consider a semiorde.

Under the point of view of the applicability of our novel approach to parabolic problems, this is highlighted in the Remarks 3.2, 3.11 and 3.17.

The second Theorem is in the spirit of Theorem 2.9 and Remark 2.16 of [16].

Theorem 2.6. Assume that there exist numbers r_i, R_i , $i = 1, 2$ with $0 < r_i < \lfloor \psi_i \rfloor R_i$ such that

$$(2.10) \quad \inf_{\substack{(u,v) \in C \\ \lfloor u \rfloor = r_1, \lfloor v \rfloor \geq r_2}} \lfloor N_1(u, v) \rfloor > r_1, \quad \inf_{\substack{(u,v) \in C \\ \lfloor u \rfloor \geq r_1, \lfloor v \rfloor = r_2}} \lfloor N_2(u, v) \rfloor > r_2,$$

and

$$(2.11) \quad \sup_{(u,v) \in C} |N_i(u, v)| \leq R_i \quad (i = 1, 2).$$

Then N has at least one fixed point $(u, v) \in K_1 \times K_2$ such that $|u| \leq R_1$, $|v| \leq R_2$ and $\lfloor u \rfloor > r_1$, $\lfloor v \rfloor > r_2$.

Sketch of the proof. The proof is similar to the proof of Theorem 2.4 and [16, Theorem 2.9]. As before, the assumption (2.11) implies that $N(C) \subset C$. Thus, $\text{ind}_C(N, C) = 1$. In order to finish the proof, it is sufficient to show that $\text{ind}_C(N, V) = 0$, where

$$V := \{(u, v) \in C : \lfloor u \rfloor < r_1 \text{ or } \lfloor v \rfloor < r_2\}.$$

We have $\partial V = B_1 \cup B_2$, where

$$\begin{aligned} B_1 &= \{(u, v) \in C : \lfloor u \rfloor = r_1, \lfloor v \rfloor \geq r_2\}, \\ B_2 &= \{(u, v) \in C : \lfloor u \rfloor \geq r_1, \lfloor v \rfloor = r_2\}. \end{aligned}$$

By (2.10) we obtain N has no fixed points on ∂V . Consider the same homotopy as in the proof of Theorem 2.2, that is

$$H(u, v, t) = th + (1 - t)N(u, v), \quad \text{where } h = (R_1\psi_1, R_2\psi_2).$$

As before we can prove that H is fixed point free on ∂V . Therefore $\text{ind}_C(N, V) = \text{ind}_C(h, V) = 0$, since, like in the previous proof, $h \notin \bar{V}$. \square

The result, in the spirit of Lemma 4 of [17], allows different types of growth of the operators and is a modification of Theorem 2.4 of [16].

Theorem 2.7. *Assume that there exist numbers r_i, R_i with $0 < r_i < \lfloor \psi_i \rfloor R_i$ such that*

$$(2.12) \quad \sup_{(u,v) \in C} |N_i(u, v)| \leq R_i \quad (i = 1, 2),$$

and

$$(2.13) \quad \inf_{(u,v) \in A} \lfloor N_1(u, v) \rfloor \geq r_1 \quad \text{or} \quad \inf_{(u,v) \in A} \lfloor N_2(u, v) \rfloor \geq r_2,$$

where A is a subset of the set

$$U = \{(u, v) \in C : \lfloor u \rfloor < r_1, \lfloor v \rfloor < r_2\}.$$

Then N has at least one fixed point $(u, v) \in K_1 \times K_2$ such that $|u| \leq R_1$, $|v| \leq R_2$ and $(u, v) \notin A$.

Proof. Since N is a completely continuous mapping in the bounded closed convex set C , by Schauder's fixed point theorem, it possesses a fixed point $(u, v) \in C$. We now show that the fixed point is not in A . Suppose on the contrary that $(u, v) = N(u, v)$ and $(u, v) \in A$. Suppose that the first inequality from (2.13) is satisfied. Then

$$r_1 > \lfloor u \rfloor = \lfloor N_1(u, v) \rfloor \geq r_1,$$

which is impossible. Similarly we arrive at a contradiction, if the second inequality from (2.13) is satisfied. \square

2.2. Multiplicity results. We present now some multiplicity results that are analogues of the results of Subsection 2.3 of [16].

Theorem 2.8. *Assume that there exist numbers ρ_i, r_i, R_i with*

$$(2.14) \quad 0 < \lfloor \psi_i \rfloor \rho_i < r_i < \lfloor \psi_i \rfloor R_i \quad (i = 1, 2),$$

such that

$$(2.15) \quad \inf_{\substack{(u,v) \in C \\ \lfloor u \rfloor = r_1, \lfloor v \rfloor \geq r_2}} \lfloor N_1(u, v) \rfloor > r_1, \quad \inf_{\substack{(u,v) \in C \\ \lfloor u \rfloor \geq r_1, \lfloor v \rfloor = r_2}} \lfloor N_2(u, v) \rfloor > r_2,$$

$$(2.16) \quad \sup_{(u,v) \in C} |N_i(u, v)| \leq R_i \quad (i = 1, 2),$$

and

$$(2.17) \quad N(u, v) \neq \lambda(u, v), \quad \text{for } \lambda \geq 1 \text{ and } (|u| = \rho_1, |v| \leq \rho_2 \text{ or } |u| \leq \rho_1, |v| = \rho_2).$$

Then N has at least three fixed points $(u_i, v_i) \in C$ ($i = 1, 2, 3$) with

$|u_1| < \rho_1$, $|v_1| < \rho_2$ (possibly zero solution);

$|u_2| < r_1$ or $|v_2| < r_2$; $|u_2| > \rho_1$ or $|v_2| > \rho_2$ (possibly one solution component zero);

$|u_3| > r_1$, $|v_3| > r_2$ (both solution components nonzero).

Proof. Let U, V be as in the proof of Theorems 2.2 and 2.6. Strict inequalities in (2.15) guarantee that N is fixed point free on ∂V . According to the proof of Theorem 2.6 we have $\text{ind}_C(N, C) = 1$, $\text{ind}_C(N, V) = 0$ and therefore by the additivity property, $\text{ind}_C(N, C \setminus \overline{V}) = 1$. Let

$$W := \{(u, v) \in C : |u| < \rho_1, |v| < \rho_2\}.$$

From (2.3), for every $(u, v) \in \overline{W}$, we have

$$|u| \leq \lfloor \psi_1 \rfloor |u| \leq \lfloor \psi_1 \rfloor \rho_1 < r_1$$

and, similarly, $|v| < r_2$. Hence $(u, v) \in U$, which proves that $\overline{W} \subset U \subset V$. Condition (2.17) shows that N is homotopic with zero on W . Thus $\text{ind}_C(N, W) = \text{ind}_C(0, W) = 1$. Then $\text{ind}_C(N, V \setminus \overline{W}) = 0 - 1 = -1$. Consequently, there exist at least three fixed points of N , in W , $V \setminus \overline{W}$ and $C \setminus \overline{V}$. \square

If we assume the following estimates of $\lfloor N_i(u, v) \rfloor$:

$$(2.18) \quad \inf_{\substack{(u,v) \in C \\ |u|=r_1}} \lfloor N_1(u, v) \rfloor > r_1, \quad \inf_{\substack{(u,v) \in C \\ |v|=r_2}} \lfloor N_2(u, v) \rfloor > r_2,$$

then we can obtain a more precise localization for the solution (u_2, v_2) in Theorem 2.8, the Figure 1 (analogous to Figure 1 of [16]) illustrates this fact.

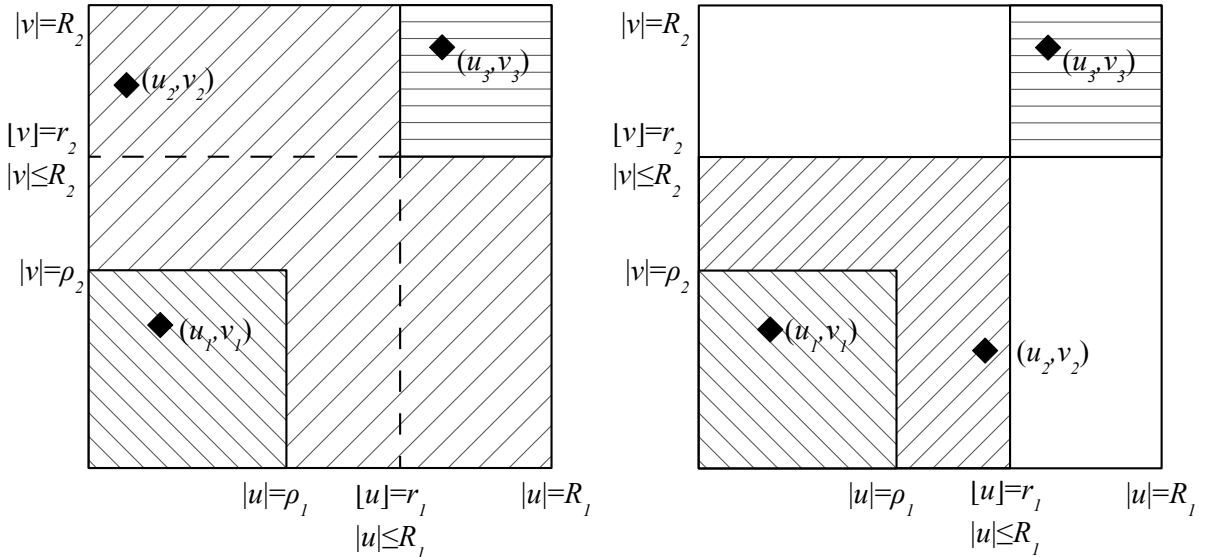


FIGURE 1. Localization of the three solutions (u_i, v_i) from Theorem 2.8 (on the left) and Theorem 2.9 (on the right).

Theorem 2.9. *Suppose that all the assumptions of Theorem 2.8 are satisfied with the condition (2.15) replaced by (2.18). Then N has at least three fixed points $(u_i, v_i) \in C$ ($i = 1, 2, 3$) with*

$$\begin{aligned} |u_1| &< \rho_1, \quad |v_1| < \rho_2 \quad (\text{possibly zero solution}); \\ \lfloor u_2 \rfloor &< r_1, \quad \lfloor v_2 \rfloor < r_2; \quad |u_2| > \rho_1 \text{ or } |v_2| > \rho_2 \quad (\text{possibly one solution component zero}); \\ \lfloor u_3 \rfloor &> r_1, \quad \lfloor v_3 \rfloor > r_2 \quad (\text{both solution components nonzero}). \end{aligned}$$

Proof. The assumption (2.18) implies both (2.4) and (2.10) and that there are no fixed points of N on ∂U and ∂V . Hence, as in the proofs of Theorems 2.2 and 2.6, the indices $\text{ind}_C(N, U)$ and $\text{ind}_C(N, V)$ are well defined and equal 0. An analysis similar to that in the proof of Theorem 2.8 shows that

$$\text{ind}_C(N, W) = 1, \quad \text{ind}_C(N, U \setminus \overline{W}) = -1, \quad \text{ind}_C(N, C \setminus \overline{V}) = 1,$$

which completes the proof. \square

In order to ensure that the solution (u_1, v_1) from the theorems above is nonzero, and thereby to obtain three *nonzero* solutions, we use some additional assumptions on N .

Theorem 2.10. *Assume that all the conditions of Theorem 2.8 or Theorem 2.9 are satisfied. Consider $0 < \varrho_i < \lfloor \psi_i \rfloor \rho_i$, $i = 1, 2$.*

(i) *If $N_1(0, 0) \neq 0$ or $N_2(0, 0) \neq 0$, then the solution (u_1, v_1) from Theorem 2.8 or 2.9 is nonzero.*

(ii) *If*

$$(2.19) \quad \inf_{\substack{(u,v) \in K, |u| \leq \rho_1, |v| \leq \rho_2 \\ \lfloor u \rfloor = \varrho_1, \lfloor v \rfloor \geq \varrho_2}} \lfloor N_1(u, v) \rfloor \geq \varrho_1, \quad \inf_{\substack{(u,v) \in K, |u| \leq \rho_1, |v| \leq \rho_2 \\ \lfloor u \rfloor \geq \varrho_1, \lfloor v \rfloor = \varrho_2}} \lfloor N_2(u, v) \rfloor \geq \varrho_2$$

and

$$|N_i(u, v)| \leq \rho_i \quad \text{for } |u| \leq \rho_1, \quad |v| \leq \rho_2 \quad (i = 1, 2),$$

then we can assume that the solution (u_1, v_1) from Theorem 2.8 or Theorem 2.9 satisfies $\lfloor u_1 \rfloor \geq \varrho_1$ and $\lfloor v_1 \rfloor \geq \varrho_2$;

(iii) *if*

$$(2.20) \quad \inf_{\substack{(u,v) \in K, |u| \leq \tilde{\rho}_1, |v| \leq \tilde{\rho}_2 \\ \lfloor u \rfloor < \varrho_1, \lfloor v \rfloor < \varrho_2}} \lfloor N_1(u, v) \rfloor \geq \varrho_1 \quad \text{or} \quad \inf_{\substack{(u,v) \in K, |u| \leq \tilde{\rho}_1, |v| \leq \tilde{\rho}_2 \\ \lfloor u \rfloor < \varrho_1, \lfloor v \rfloor < \varrho_2}} \lfloor N_2(u, v) \rfloor \geq \varrho_2$$

for some $\tilde{\rho}_1 \leq \rho_1$, $\tilde{\rho}_2 \leq \rho_2$, then we can assume that the solution (u_1, v_1) from Theorem 2.8 or Theorem 2.9 satisfies $\lfloor u_1 \rfloor \geq \varrho_1$ or $\lfloor v_1 \rfloor \geq \varrho_2$ or $|u_1| > \tilde{\rho}_1$ or $|v_1| > \tilde{\rho}_2$.

Proof. (i) The assumption implies that $(0, 0)$ is not a fixed point.

(ii) The inequality follows from Theorem 2.6 applied in the case of $r_i := \varrho_i$ and $R_i := \rho_i$.

(iii) From Theorem 2.7 applied in the case of $r_i := \varrho_i$, $R_i := \rho_i$ and

$$A = \{(u, v) : \lfloor u \rfloor < \varrho_1, \lfloor v \rfloor < \varrho_2, |u| \leq \tilde{\rho}_1, |v| \leq \tilde{\rho}_2\},$$

we obtain there are no fixed points of N in A , which ends the proof. \square

The next Remark illustrates how Theorem 2.6 can be used to prove the existence of more nontrivial solutions.

Remark 2.11. If N satisfies the conditions of Theorem 2.6 for all pairs

$$0 < r_i^j < \lfloor \psi_i \rfloor R_i^j \text{ for } i = 1, 2, j = 1, 2, \dots, n,$$

satisfying

$$\lfloor \psi_i \rfloor R_i^j < r_i^{j+1} \text{ for } i = 1, 2, j = 1, 2, \dots, n-1,$$

then N possesses at least n nontrivial solutions (u_j, v_j) with

$$|u_j| \leq R_1^j, |v_j| \leq R_2^j, \lfloor u_j \rfloor > r_1^j, \lfloor v_j \rfloor > r_2^j.$$

Moreover, if (2.11) holds with the strict inequality, i.e. if

$$\sup_{(u,v) \in K, |u| \leq R_1^j, |v| \leq R_2^j} |N_i(u, v)| < R_i^j \quad (i = 1, 2),$$

hold, then we have $n-1$ additional solutions (\bar{u}_j, \bar{v}_j) , $j = 1, \dots, n-1$ such that

$$|\bar{u}_j| < R_1^{j+1}, |\bar{v}_j| < R_2^{j+1}; |\bar{u}_j| > R_1^j \text{ or } |\bar{v}_j| > R_2^j; \lfloor \bar{u}_j \rfloor < r_1^{j+1} \text{ or } \lfloor \bar{v}_j \rfloor < r_2^{j+1}.$$

The first conclusion follows from Theorem 2.6 applied n times, whereas the second follows from Theorem 2.8 applied $n-1$ times.

Remark 2.12. We stress that the abstract results obtained in this section can be generalized to the case of systems of more than two equations. The idea is to consider the product space $E = \Pi_{i=1}^n E_i$ of the Banach spaces E_i , endowed with the norms $|\cdot|_i$, functionals $\lfloor \cdot \rfloor_i$, and the pairs of cones and wedges $K_i \subset G_i \subset E_i$ such that (2.1), (2.2) are satisfied for $i = 1, 2, \dots, n$. In this setting we are interested in the existence and localization of fixed points of a given operator $N: K \rightarrow K$, where $K = \Pi_{i=1}^n K_i$. For example, let us consider the sets

$$C = \{u \in K : |u_1|_1 \leq R_1, \dots, |u_n|_n \leq R_n\}, \quad U = \{u \in C : \lfloor u_1 \rfloor_1 < r_1, \dots, \lfloor u_n \rfloor_n < r_n\}$$

for given radii $r_i, R_i > 0$ with $r_i < \lfloor \psi_i \rfloor_i R_i$, $i = 1, \dots, n$. If

$$\sup_{u \in C} |N_i(u)|_i \leq R_i, \quad i = 1, 2, \dots, n$$

and

$$\inf_{\substack{u \in \bar{U} \\ \lfloor u_i \rfloor_i = r_i}} \lfloor N_i(u) \rfloor_i > r_i, \quad i = 1, 2, \dots, n,$$

then N has at least one fixed point in $C \setminus \bar{U}$.

As a consequence, results analogous to ones obtained later in Section 3, can be established for systems with more than two differential equations.

3. THE SYSTEM OF PARABOLIC EQUATIONS

Let $\Omega \subset \mathbb{R}^m$ be a bounded domain that is Dirichlet regular (i.e. for all $\varphi \in C(\partial\Omega)$ there exists $u \in C(\bar{\Omega})$ such that $u|_{\partial\Omega} = \varphi$ and $\Delta u = 0$ in a distributional sense, see [1, Definition 6.1.1]). This class of domains is rather large; for example, if the boundary of Ω is Lipschitz continuous, then Ω is Dirichlet regular (see [8, Chapter II, Section 4, Proposition 4]).

Let us take

$$E = C_0(\Omega) = \{u \in C(\overline{\Omega}) : u|_{\partial\Omega} = 0\}, \quad E_+ = \{u \in E : u(x) \geq 0 \text{ for all } x \in \Omega\}.$$

Let us also consider the space $\mathcal{E} = C(0, t_{\max}, E)$, $t_{\max} > 0$ and its cone of nonnegative functions $\mathcal{E}_+ = C(0, t_{\max}, E_+)$. The spaces E and \mathcal{E} are endowed with the uniform norms, that is

$$|u| = \max \{|u(x)| : x \in \overline{\Omega}\}, \quad u \in E,$$

and

$$|u| = \max \{|u(t)| : 0 \leq t \leq t_{\max}\}, \quad u \in \mathcal{E}.$$

We shall discuss the parabolic system

$$(3.1) \quad \begin{cases} u_t - \Delta u = f(t, x, u, v) & (t, x) \in Q := (0, t_{\max}) \times \Omega, \\ v_t - \Delta v = g(t, x, u, v) & (t, x) \in Q := (0, t_{\max}) \times \Omega, \\ u(t, x) = v(t, x) = 0 & (t, x) \in (0, t_{\max}) \times \partial\Omega, \\ u(0, \cdot) = \alpha(u, v), \\ v(0, \cdot) = \beta(u, v). \end{cases}$$

Here, $f, g: (0, t_{\max}) \times \Omega \times \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and $\alpha, \beta: \mathcal{E} \times \mathcal{E} \rightarrow E$ are continuous functions. In what follows we shall identify $u: [0, t_{\max}] \times \overline{\Omega} \rightarrow \mathbb{R}$ with $u: [0, t_{\max}] \rightarrow C(\overline{\Omega})$ via the formula $u(t)(x) = u(t, x)$.

We shall treat Δ as the operator defined on the domain $D(\Delta) = \{u \in E : \Delta u \in E\}$.

Lemma 3.1. *The operator Δ is a generator of an analytic (immediately) compact C_0 -semigroup of contractions $\{S(t) : t \geq 0\}$ on E . Moreover, the operators $S(t)$ are positive, i.e. $S(t)u \geq 0$ for $u \geq 0$.*

Proof. From [1, Theorem 6.1.8] we conclude that the semigroup S is positive and contractive. From [1, Theorem 6.1.9] we see that S is analytic. According to Theorems 4.29 and 4.25 of [11], to demonstrate the compactness of S it remains to prove the compactness of the canonical injection $i: E_\Delta \hookrightarrow E$, where $E_\Delta = (D(\Delta), \|\cdot\|_\Delta)$ is equipped with the graph norm $\|\cdot\|_\Delta$.

Let us consider the Fréchet space $F = E \cap C^1(\Omega)$ equipped with the seminorms p_n ,

$$p_0(u) = \max \{|u(x)| : x \in \Omega\}, \quad p_n(u) = \max \{|\nabla u(x)| : x \in \overline{\Omega_n}\}, \quad n \geq 1,$$

where (Ω_n) is an open covering of Ω satisfying $\overline{\Omega_n} \subset \Omega$, $n \in \mathbb{N}$.

By [1, Lemma 6.1.5] we know that $D(\Delta) \subset F$. The Closed Graph Theorem yields the embedding $E_\Delta \hookrightarrow F$ is continuous. The conclusion is implied by the compactness of $F \hookrightarrow E$. \square

Remark 3.2. It seems worth discussing the choice of the space $C_0(\Omega)$. In the recent paper [16], where an elliptic system was discussed, the space $L^\infty(\Omega)$ was considered. Unfortunately, the Laplacian Δ fails to generate a C_0 -semigroup on $L^\infty(\Omega)$. Moreover, although Δ generates semigroups on $L^p(\Omega)$, these spaces are somewhat inappropriate to obtain the localization of solutions with our approach. The fact that the set $\{u \in E_+ : |u| \leq 1\}$

does not possess a largest element (with respect to the natural order \leq) prevented us from using the abstract setting from [16]; for details, see Remark 3.11.

We shall also consider the space $H = L^2(\Omega)$ and the Laplacian Δ_2 on H with Dirichlet boundary condition. Denote by $S_2: [0, \infty) \rightarrow B(H)$ the semigroup generated by Δ_2 and by $i: E \rightarrow H$ the natural embedding.

Proposition 3.3. $i(S(t)u) = S_2(t)i(u)$ for all $u \in E$.

Proof. Since $D(\Delta_2) = \{u \in H : \Delta u \in H\}$, Δ is the part of Δ_2 in E (see [11, Paragraph II.2.3]) and, as a consequence, $\Delta_2 \circ i = i \circ \Delta$. This implies that $R(\lambda, \Delta_2) \circ i = i \circ R(\lambda, \Delta)$ for $\lambda > 0$. Using the Post-Widder inversion formula for C_0 -semigroups (see Corollary 3.3.6 in [1]) we obtain for $x \in E$ and $t \geq 0$

$$i(S(t)x) = \lim_{n \rightarrow \infty} i \left(\left[\frac{n}{t} R \left(\frac{n}{t}, \Delta \right) \right]^n x \right) = \lim_{n \rightarrow \infty} \left[\frac{n}{t} R \left(\frac{n}{t}, \Delta_2 \right) \right]^n i(x) = S_2(t)i(x).$$

□

Define

$$F(u, v)(t)(x) = f(t, x, u(t)(x), v(t)(x)), \quad G(u, v)(t)(x) = g(t, x, u(t)(x), v(t)(x)).$$

Under the following assumption:

$$(3.2) \quad f(t, x, 0, 0) = g(t, x, 0, 0) = 0 \quad \text{for } x \in \partial\Omega,$$

the operators $F, G: \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{E}$ are continuous and bounded (map bounded sets into bounded ones).

Definition 3.4. For $\xi \in E$, and $f \in \mathcal{E}$, we say that the function

$$u(t) = p(\xi, f)(t) := S(t)\xi + \int_0^t S(t-s)f(s)ds$$

is a *mild solution* of the problem $u' - \Delta u = f$ on $(0, t_{\max}) \times \Omega$ with $u(0) = \xi$.

Therefore, the mild solution of the problem (3.1) is a fixed point of the vector valued operator M defined as

$$M_1(u, v) = \bar{S}(\alpha(u, v)) + \hat{S}(F(u, v)), \quad M_2(u, v) = \bar{S}(\beta(u, v)) + \hat{S}(G(u, v)),$$

where

$$\begin{aligned} \bar{S}: E &\rightarrow \mathcal{E}, \quad \bar{S}(u_0)(t) = S(t)u_0, \quad u_0 \in E, \\ \hat{S}: \mathcal{E} &\rightarrow \mathcal{E}, \quad \hat{S}(f)(t) = \int_0^t S(t-\tau)f(\tau)d\tau, \quad f \in \mathcal{E}. \end{aligned}$$

It is worth pointing out the following regularity of the mild solutions.

Proposition 3.5. Let $u = p(\xi, f)$ for $\xi \in E$, $f \in \mathcal{E}$, that is u is a mild solution of the problem $u' - \Delta u = f$, $u(0) = \xi$. Then

- (a) u is a strong solution of that problem in the space $L^2(\Omega)$. Precisely, $u: [0, t_{\max}] \rightarrow L^2(\Omega)$ is absolutely continuous, $u' \in L^1(0, t_{\max}, L^2(\Omega))$, $u(t) \in D(\Delta_2)$ and $u'(t) = \Delta_2 u(t) + f(t)$ for almost all $t \in (0, t_{\max})$.

(b) u is a weak solution of the equation $u' - \Delta_2 u = f$ in the sense that the weak spatial derivative $\nabla u(t, x)$ and weak time derivative $u_t(t, x)$ exist on $(0, t_{\max}) \times \Omega$ and

$$(3.3) \quad \int_0^{t_0} \int_{\Omega} \nabla u(t, x) \nabla \phi(t, x) - u \frac{\partial}{\partial t} \phi(t, x) dx dt = \int_0^{t_0} \int_{\Omega} f(t) \phi(t, x) dt$$

for all $\phi \in C_0^\infty((0, t_{\max}) \times \Omega)$.

Proof. (a) Let u be as in the statement.

$$\bar{u}(t) = S_2(t)\bar{\xi} + \int_0^t S_2(t-s)\bar{f}(s)ds, \text{ where } \bar{u} = i \circ u, \bar{\xi} = i(\xi), \bar{f} = i \circ f.$$

The conclusion follows from [27, Theorem 8.2.1], as $H = L^2(\Omega)$ is a Hilbert space.

(b) We now prove the following statement: The weak spatial derivative $\nabla u(t, x)$ exists on $(0, t_{\max}) \times \Omega$ and belongs to $L^2((0, t_{\max}) \times \Omega)$.

Because $u(t) \in D(\Delta_2) \subset W_0^{1,2}(\Omega)$ for $t \in (0, t_{\max})$, the weak (spatial) derivative $\nabla u(t)$ exists for almost all $t \in (0, t_{\max})$. Therefore we have

$$(3.4) \quad \int_0^{t_0} \int_{\Omega} u(t, x) \nabla \phi(t, x) dx dt = - \int_0^{t_0} \int_{\Omega} \nabla(u(t))(x) \phi(t, x) dx dt,$$

for all $\phi \in C_0^\infty((0, t_{\max}) \times \Omega)$, as $\phi(t, \cdot) \in C_0^\infty(\Omega)$ for all $t \in (0, t_{\max})$. Thus, the weak spatial derivative $\nabla u(t, x)$ exists and is equal to $\nabla(u(t))(x)$. Moreover,

$$(3.5) \quad \|\nabla u\|_{L^2((0, t_{\max}) \times \Omega)}^2 = \int_0^{t_0} \int_{\Omega} |\nabla u(t, x)|^2 dx dt = \int_0^{t_0} \|\nabla(u(t))\|_H^2 dt.$$

By [27, Theorem 8.2.1] we know that the function

$$t \mapsto -\langle \Delta_2 u(t), u(t) \rangle_H = \int_{\Omega} |(\nabla u(t))(x)|^2 dx = \|\nabla(u(t))\|_H^2$$

belongs to $L^1(0, t_{\max})$. This and (3.5) prove that $\|\nabla u\|_{L^2((0, t_{\max}) \times \Omega)}^2 < \infty$.

We now prove that the weak time derivative $u_t(t, x)$ exists on $(0, t_{\max}) \times \Omega$.

Let us consider the function $\phi \in C_0^\infty((0, t_{\max}) \times \Omega)$. To avoid disambiguity let us introduce the function $\varphi: (0, t_{\max}) \rightarrow C_0(\Omega)$ defined by the formula $\varphi(t) = \phi(\cdot, t)$. One can easily check that $\varphi \in C^1(0, t_{\max}, C_0(\Omega))$ and $\varphi'(t) = \phi_t(\cdot, t)$.

By the absolute continuity of $u, \varphi: (0, t_{\max}) \rightarrow H$ we obtain $d(t) = \langle u(t), \varphi(t) \rangle_H$ is absolutely continuous and $d'(t) = \langle u'(t), \varphi(t) \rangle_H + \langle u(t), \varphi'(t) \rangle_H$. In particular,

$$(3.6) \quad 0 = d(t_0) - d(0) = \int_0^{t_0} d'(t) dt = \int_0^{t_0} \langle u'(t), \varphi(t) \rangle_H + \langle u(t), \varphi'(t) \rangle_H dt.$$

Since $u' \in L^1(0, t_{\max}, H)$ and $u, \varphi, \varphi' \in L^\infty(0, t_{\max}, H)$, both functions $\langle u'(t), \varphi(t) \rangle_H$, $\langle u(t), \varphi'(t) \rangle_H$ are integrable. Thus, we can restate (3.6) in the following manner:

$$(3.7) \quad \int_0^{t_0} \int_{\Omega} u(t, x) \phi_t(t, x) dx dt = - \int_0^{t_0} \int_{\Omega} (u'(t))(x) \phi(t, x) dx dt.$$

This proves that the weak derivative $\frac{\partial}{\partial t} u(t, x)$ exists and is equal to $(u'(t))(x)$.

Since $u(t) \in D(\Delta_2)$ for a.a. $t \in (0, t_{\max})$ and $\nabla u(t, x) = \nabla(u(t))(x)$, we have

$$(3.8) \quad \int_0^{t_0} \int_{\Omega} \Delta_2 u(t)(x) \phi(t, x) dx dt = - \int_0^{t_0} \int_{\Omega} \nabla u(t, x) \nabla \phi(t, x) dx dt.$$

Let $\phi \in C_0^\infty((0, t_{\max}) \times \Omega)$. Combining the equations (3.8), (3.7) and $u'(t) = \Delta_2 u(t) + f(t)$ for almost all $t \in (0, t_{\max})$ (see (a)) we obtain (3.3). \square

Proposition 3.6. *The operators M_1, M_2 are completely continuous if and only if α and β are completely continuous.*

Proof. If M_1, M_2 are completely continuous, then $e_0 \circ M_1 = \alpha, e_0 \circ M_2 = \beta$ are completely continuous, where $e_0(f) = f(0)$ for $f \in \mathcal{E}$.

Now, assume that the operators α, β are completely continuous. Since the operator \bar{S} is continuous, then the operator $\bar{S} \circ \alpha$ is completely continuous. We shall demonstrate that \hat{S} is completely continuous. In order to do this we utilize a version of the Ascoli-Arzelà Theorem tailored for the space \mathcal{E} , see for example Theorem A.2.1 of [27].

First, denote by Ξ the upper bound of the norms of $\|S(t)\|_{B(E)}$ for $t \in [0, t_{\max}]$. Because S is immediate norm continuous, for any $\varepsilon > 0$ there exists a number $\mu(\varepsilon) > 0$ such that $\|S(t) - S(s)\|_{B(E)} < \varepsilon$ if $0 < \varepsilon \leq t \leq s \leq t + \mu(\varepsilon)$.

Fix $R, \varepsilon > 0$ and let $D = \{f \in \mathcal{E} : |f| < R\}$. For a fixed $t > 0$ and $f \in D$ let $\eta = \min\{t, \varepsilon\}$. We can present $\hat{S}(f)(t)$ as a sum $\hat{S}(t) = x + y$, where

$$x = S(\eta) \int_0^{t-\eta} S(t-s-\eta) f(s) ds, \quad y = \int_{t-\eta}^t S(t-s) f(s) ds.$$

It is straightforward to show that $x \in \hat{C} := S(\eta)(B(0, t\Xi R))$ and that $y \in B(0, \varepsilon\Xi R)$. Because \hat{C} is relatively compact and ε is arbitrary, we obtain the set $\{\hat{S}(f)(t) : f \in D\}$ is relatively compact for all $t \in [0, t_{\max}]$.

Now we shall prove the equicontinuity of the family $\{\hat{S}(f) : f \in D\}$. In order to do it let us fix $\varepsilon > 0$, $t, s \in [0, t_{\max}]$ and $f \in D$. Without loss of generality we can assume that $t \leq s$. Let us put $\eta = \min\{t, \varepsilon\}$ (the case $\eta = 0$ appears if $t = 0$). Then $\hat{S}(f)(s) - \hat{S}(f)(t) = x + y$, where

$$x = \int_0^t (S(s-\tau) f(\tau) - S(t-\tau) f(\tau)) d\tau, \quad y = \int_t^s S(s-\tau) f(\tau) d\tau.$$

$$|x| \leq R \int_0^t \|S(s-t+\tau) - S(\tau)\|_{B(E)} d\tau \leq 2\Xi R\varepsilon + \int_\eta^t \|S(s-t+\tau) - S(\tau)\|_{B(E)} d\tau.$$

$$|y| \leq |s-t|\Xi R.$$

Therefore $|\hat{S}(f)(s) - \hat{S}(f)(t)| \leq (3\Xi R + t_{\max})\varepsilon$ if $|s-t| \leq \min\{\mu(\varepsilon), \varepsilon\}$. This proves the uniform equicontinuity of the family $\{\hat{S}(f) : f \in D\}$ and finishes the proof of the complete continuity of \hat{S} . Now, the complete continuity of $M_1 = \bar{S} \circ \alpha + \hat{S} \circ F$ is clear. Similarly we can prove the complete continuity of M_2 . \square

In order to use fixed point index for compact operators and, at the same time, to avoid assuming the compactness of α and β , we consider the operator $N = (N_1, N_2)$, defined

by the formula

$$\begin{aligned} N_1(u, v) &= \bar{S} \left(\alpha \left(\bar{S}(u(0)) + \hat{S}(F(u, v)) \right), \bar{S}(v(0)) + \hat{S}(G(u, v)) \right) + \hat{S}(F(u, v)), \\ N_2(u, v) &= \bar{S} \left(\beta \left(\bar{S}(u(0)) + \hat{S}(F(u, v)) \right), \bar{S}(v(0)) + \hat{S}(G(u, v)) \right) + \hat{S}(G(u, v)). \end{aligned}$$

Proposition 3.7. *The sets of fixed points of the operators M and N coincide.*

Proof. Note that

$$(3.9) \quad N_1(u, v) = \bar{S}(u_0) + \hat{S}(F(u, v)) \text{ and } N_2(u, v) = \bar{S}(v_0) + \hat{S}(G(u, v)),$$

where $u_0 = \alpha(\bar{u}, \bar{v})$, $v_0 = \beta(\bar{u}, \bar{v})$ and

$$(3.10) \quad \bar{u} = \bar{S}(u(0)) + \hat{S}(F(u, v)), \quad \bar{v} = \bar{S}(v(0)) + \hat{S}(G(u, v)).$$

From (3.9) and the properties of \bar{S} and \hat{S} we have $N_1(u, v)(0) = u_0$ and $N_2(u, v)(0) = v_0$. Therefore, if $N(u, v) = (u, v)$, then $\bar{u} = u$ and $\bar{v} = v$ and, consequently, $u_0 = \alpha(u, v)$ and $v_0 = \beta(u, v)$. By (3.9) we arrive at $M(u, v) = (u, v)$.

Conversely, if $M(u, v) = (u, v)$, then

$$(3.11) \quad u = \bar{S}(\alpha(u, v)) + \hat{S}(F(u, v)), \quad v = \bar{S}(\beta(u, v)) + \hat{S}(G(u, v))$$

and $u(0) = \alpha(u, v)$, $v(0) = \beta(u, v)$. Therefore we have

$$u = \bar{S}(u(0)) + \hat{S}(F(u, v)) \text{ and } v = \bar{S}(v(0)) + \hat{S}(G(u, v)).$$

Plugging this into (3.11) we obtain $N(u, v) = (u, v)$. □

From the proof it follows in particular, that

$$(3.12) \quad \begin{aligned} N_1(u, v) &= \bar{S}(N_1(u, v)(0)) + \hat{S}(F(u, v)), \text{ and} \\ N_2(u, v) &= \bar{S}(N_2(u, v)(0)) + \hat{S}(G(u, v)), \end{aligned}$$

which will be used later.

From Proposition 3.6 we know that a necessary condition for the operator M to be completely continuous is the complete continuity of α and β . In the case of the operator N we can weaken the assumptions on α and β .

Proposition 3.8. *The operator N is completely continuous if the images $\alpha(U_1 \times U_2)$ and $\beta(U_1 \times U_2)$ are relatively compact for all bounded sets $U_1, U_2 \subset \mathcal{E}$ that satisfy the following property:*

$$(3.13) \quad \text{For all } \varepsilon > 0 \text{ the set } \{u(t) : u \in U_i, t \in [\varepsilon, t_{\max}]\} \text{ is relatively compact, } i = 1, 2.$$

Proof. We prove, without loss of generality, the complete continuity of N_1 . Let $\bar{\alpha}(u, v) = \alpha(\bar{u}, \bar{v})$, where $\bar{u}(u, v)$ and $\bar{v}(u, v)$ are defined by (3.10). From (3.9) it follows that $N_1(u, v) = \bar{S}(\bar{\alpha}(u, v)) + \hat{S}(F(u, v))$. By Proposition 3.6, it suffices to show that $\bar{\alpha}$ is completely continuous. This will be done if we demonstrate, that $\bar{u}(U \times U)$ and $\bar{v}(U \times U)$ satisfy (3.13) for a given bounded set $U \subset \mathcal{E}$.

Let $\varepsilon > 0$ be given. Put $R = \sup \{|u| : u \in U\}$. For $u, v \in U$ and $t \geq \varepsilon$ we obtain

$$\bar{S}(u(0))(t) = S(\varepsilon)S(t - \varepsilon)u(0) \in S(\varepsilon)D(0, R) =: \mathcal{C}$$

and the set \mathcal{C} is relatively compact. Moreover, the proof of Proposition 3.6 shows that the set $\hat{S}(F(U \times U))$ is relatively compact in \mathcal{E} . By the standard arguments, utilizing the compactness of $[0, t_{\max}]$, one can show that the set

$$\left\{ \hat{S}(F(u, v))(t) : u, v \in U, t \in [0, t_{\max}] \right\}$$

is totally bounded (and therefore relatively compact) in E . This shows that $\bar{u}(U \times U)$ satisfies the condition (3.13). Similarly we can verify this condition for the set $\bar{v}(U \times U)$. \square

Example 3.9. Let $\alpha(u, v) = u(t_0)$, $\beta(u, v) = v(t_0)$, where $0 \leq t_0 \leq t_{\max}$. Then α, β satisfy the condition from Proposition 3.8 if and only if $t_0 > 0$. Indeed, let $t_0 = 0$ and $U_1 = U_2 = \{S(\cdot)u : |u| \leq 1\}$. Then the condition (3.13) is satisfied, but the set

$$\alpha(U_1 \times U_2) = \{u(0) : u \in U_1\} = \{u \in E : |u| \leq 1\}$$

is not compact.

Conversely, if $t_0 > 0$ and sets U_1, U_2 satisfy the condition (3.13), then $\alpha(U_1 \times U_2) = \{u(t_0) : u \in U_1\}$ is compact from (3.13).

Note that the reasoning above can be applied to the case of *multi-point* conditions of the type

$$\alpha(u, v) = \sum_{s=1}^k \alpha_s u(t_s), \quad \beta(u, v) = \sum_{s=1}^r \beta_s v(t'_s),$$

where $0 < t_1 < \dots < t_k$, $0 < t'_1 < \dots < t'_r$ and $\alpha_s, \beta_s > 0$.

Example 3.10. Let

$$\alpha(u, v) = G_1 \left(\int_0^{t_{\max}} g_1(u(t), v(t)) dt \right), \quad \beta(u, v) = G_2 \left(\int_0^{t_{\max}} g_2(u(t), v(t)) dt \right),$$

where $g_1, g_2 : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$, $G_1, G_2 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $g_i(0, 0) = G_i(0) = 0$ for $i \in \{1, 2\}$. Then α, β satisfy the condition from Proposition 3.8.

Indeed, let us consider the sets $U_1, U_2 \subset B(0, R)$ satisfying the condition (3.13) and let $\varepsilon > 0$. From the uniform continuity of G_i on $[0, t_{\max} \cdot d]$, where

$$d = \max \{g_i(u, v) : 0 \leq u, v \leq R, i = 1, 2\},$$

there exists $\delta > 0$ such that $|G(p) - G(q)| < \varepsilon$ if $0 \leq p, q \leq t_{\max} \cdot d$, $|p - q| < \delta \cdot d$.

Put

$$U_i^\delta := \{u(t) : u \in U_i, t \in [\delta, t_{\max}]\}, \quad \delta > 0, \quad i = 1, 2$$

and

$$\Gamma_i^\delta := \overline{\text{conv}} \, g_i(U_1^\delta \times U_2^\delta).$$

From the Mazur Theorem, which states that the closed convex hull of a compact subset in a Banach space is compact, the sets Γ_i^δ are compact. Therefore,

$$\int_\delta^{t_{\max}} g_i(u(t), v(t)) dt \in W_i^\delta := (t_{\max} - \delta) \cdot \Gamma_i^\delta \quad \text{and} \quad \left| \int_0^\delta g_i(u(t), v(t)) dt \right| \leq \delta \cdot d$$

for $u \in U_1, v \in U_2$. Thus, by the choice of δ , we deduce that for all $u \in U_1, v \in U_2$ we have $\alpha(u, v) = u_1 + u_2$, where $u_1 \in G_1(W_1^\delta)$ and $|u_2| < \varepsilon$. Because $G_1(W_1^\delta)$ is compact

and ε is arbitrarily small, we obtain $\alpha(U_1 \times U_2)$ is relatively compact. Similarly we can proceed with β .

Let $D \subset\subset \Omega$ be any open subset. Put

$$G = \{u \in \mathcal{E} : u(0, x) \geq 0 \text{ for all } x \in D\}.$$

The set G is a wedge generating the semiorder \preceq . By \leq we denote the order induced by the cone \mathcal{E}_+ . The symbol \leq will also be used to denote the order on E induced by E_+ and the natural order on $H = L^2(\Omega)$ (that is $u \leq v$ if $u(x) \leq v(x)$ for a.a. $x \in \Omega$).

Given a function $u \in H$ we set

$$\lfloor u \rfloor = \operatorname{ess\,inf}_{x \in D} |u(x)|,$$

(in particular $\lfloor u \rfloor = \inf_{x \in D} |u(x)|$ for $u \in E$) and furthermore, with abuse of notation, by the same symbol we denote the value $\lfloor u \rfloor = \lfloor u(0) \rfloor$ for $u \in \mathcal{E}$.

The following monotonicity and continuity conditions of the functional $\lfloor \cdot \rfloor$ are satisfied:

$$\lfloor u \rfloor \leq \lfloor v \rfloor \text{ if } u, v \in \mathcal{E}_+ \text{ and } u \preceq v$$

and

$$|\lfloor u \rfloor - \lfloor v \rfloor| \leq |u - v| \text{ for all } u, v \in \mathcal{E} \quad (\text{therefore } \lfloor u \rfloor \leq |u| \text{ for all } u \in \mathcal{E}).$$

Consider the function $\psi(t) = \varphi$ for all $t \in [0, t_{\max}]$, where $\varphi \in E$ satisfies the following conditions: $\varphi|_D \equiv 1$ and $0 \leq \varphi \leq 1$ in Ω . Then $|\psi| = 1$ and $u \preceq |u|\psi$ for every $u \in \mathcal{E}_+$.

Remark 3.11. Note that in the space $\mathcal{E} = C(0, t_{\max}, C_0(\Omega))$ there is no element $\bar{\psi}$ that $|\bar{\psi}| = 1$ and $u \leq |u|\bar{\psi}$ for every $u \in \mathcal{E}_+$. This is the main reason for considering the wedges G_i , $i \in \{1, 2\}$, and the semiorders \preceq in Section 2.

Define the cones

$$K_1 = K_2 = \{u \in \mathcal{E}_+ : u(t) \geq S(t)u(0) \text{ for all } t \in [0, t_{\max}]\}.$$

We see that (3.12) yields $N(\mathcal{E}_+ \times \mathcal{E}_+) \subset K_1 \times K_2 = K$.

Proposition 3.12. Assume that $u_0 \in E$ is a nonnegative nonzero function. Define $u(t, x) = (S(t)u_0)(x)$. Then

- (i) $u \in C^\infty((0, \infty) \times \Omega) \cap C(\mathbb{R}_+ \times \bar{\Omega})$
- (ii) $u_t = \Delta u$ on $(0, \infty) \times \Omega$
- (iii) $u(0, x) = u_0(x)$ for $x \in \Omega$
- (iv) $u(t, x) > 0$ for $t > 0$ and $x \in \Omega$.

Proof. The conclusions (i)-(iii) follow from Proposition 2.6 of [1]. To prove (iv) let us fix $x \in \Omega$ and $t > 0$. Let us consider the domain $V \subset\subset \Omega$ such that $x, x_0 \in V$, where $x_0 \in \Omega$ is such that $u_0(x_0) > 0$. From (i) it follows that $\sup_V u(\varepsilon, \cdot) > 0$ for $\varepsilon < t$ sufficiently small. From (i) and (ii) it follows that u solves $u_t = \Delta u$ on $(0, \infty) \times V$. Moreover, $u \geq 0$, which follows from the positiveness of the semigroup S .

We shall use the *parabolic Harnack inequality* (see for example Theorem 7.1.10 of [12]) expressed in the following manner:

Let $u \in C^2((0, \infty) \times \Omega)$ with $u_t = \Delta u$ and $u \geq 0$ in $(0, \infty) \times \Omega$. Let $V \subset\subset \Omega$ be connected. Then, for each $0 < t_1 < t_2$, there exists a constant C such that

$$\sup_V u(t_1, \cdot) \leq C \inf_V u(t_2, \cdot).$$

From the parabolic Harnack inequality we have $\inf_V u(t, \cdot) > 0$. Therefore we obtain $u(t, x) > 0$. \square

Lemma 3.13. Let $0 \neq \eta \in E = C_0(\Omega)$.

(a) If $0 < t_0 < t_1$, then

$$m_\eta(t_0, t_1) := \min \{ (S(\tau)\eta)(x) : x \in \overline{D}, \tau \in [t_0, t_1] \} > 0.$$

(b) If $0 \leq t_0 < t_1$, then

$$\left| \int_{t_0}^{t_1} S(\tau)\eta d\tau \right| > 0.$$

Proof. (a) By Proposition 3.12(i,iv) and compactness of $[t_0, t_1] \times \overline{D}$ we obtain $m_\eta(t_0, t_1) > 0$.

(b) We can assume that $t_0 > 0$. Then

$$\int_{t_0}^{t_1} (S(\tau)\eta)(x) d\tau \geq (t_1 - t_0)m_\eta(t_0, t_1) > 0$$

for $x \in D$, which is the desired conclusion. \square

Define

$$(3.14) \quad m(t_0, t_1) := \operatorname{ess\,inf}_{x \in D, t \in [t_0, t_1]} (S_2(t)\chi_D)(x).$$

Corollary 3.14. (a) If $0 < t_0 < t_1$, then $m(t_0, t_1) > 0$.

(b) If $0 \leq t_0 \leq t_1$, then

$$\left| \int_{t_0}^{t_1} S_2(\tau)\chi_D d\tau \right| > 0.$$

It is possible that $m(0, t_1) > 0$ (see Example 3.3).

Proof. Let $\eta \in E = C_0(\Omega)$ be any nonzero function with $0 \leq \eta \leq \chi_D$. The conclusion is implied by Lemma 3.13 and the inequality $S_2(t)\chi_D \geq S_2(t)\eta = S(t)\eta$, which follows from the positiveness of S_2 and Proposition 3.3. \square

Let us fix $0 \leq t_0 < t_1 \leq t_{\max}$ and put $m := m(t_0, t_1)$.

Remark 3.15. Take $u \in K$ and set $r := \lfloor u \rfloor = \inf_{x \in D} u(0, x)$. Observe that $u(0) \geq r\chi_D$ and therefore

$$u(t) \geq S(t)u(0) = S_2(t)u(0) \geq rS_2(t)\chi_D \geq rm\chi_D \quad \text{for } t \in [t_0, t_1].$$

As a consequence, we obtain the estimate

$$(3.15) \quad u \geq m \lfloor u \rfloor \chi_{[t_0, t_1] \times D} \quad \text{for all } u \in K,$$

which can be called *weak Harnack-type inequality*, a counterpart of the inequality (3.4) of [16].

Let α, β be as in Example 3.10. Consider the assumption

$$(3.16) \quad p_1 u \leq g_1(u, v) \leq q_1 u, \quad p_2 v \leq g_2(u, v) \leq q_2 v, \quad P_i u \leq G_i(u) \leq Q_i u,$$

for all $u, v \geq 0, i \in \{1, 2\}$, where $0 < p_i \leq q_i, 0 < P_i \leq Q_i, i \in \{1, 2\}$. Similar assumptions were used, for nonlinear nonlocal conditions in the context of ODEs, in [15, 17].

Under the assumption (3.16) we have the following estimates:

$$(3.17) \quad \begin{aligned} p_1 P_1 \int_0^{t_{\max}} u(t) dt &\leq \alpha(u, v) \leq q_1 Q_1 \int_0^{t_{\max}} u(t) dt, \\ p_2 P_2 \int_0^{t_{\max}} v(t) dt &\leq \beta(u, v) \leq q_2 Q_2 \int_0^{t_{\max}} v(t) dt. \end{aligned}$$

Put

$$(3.18) \quad f_{r,R}^0 := \inf_{\substack{t \in [t_0, t_1], x \in D \\ mr_1 \leq u \leq R_1, 0 \leq v \leq R_2}} \frac{f(t, x, u, v)}{r_1}, \quad g_{r,R}^0 := \inf_{\substack{t \in [t_0, t_1], x \in D \\ 0 \leq u \leq R_1, mr_2 \leq v \leq R_2}} \frac{g(t, x, u, v)}{r_2},$$

$$(3.19) \quad f_{r,R} := \inf_{\substack{t \in [t_0, t_1], x \in D \\ mr_1 \leq u \leq R_1, mr_2 \leq v \leq R_2}} \frac{f(t, x, u, v)}{r_1}, \quad g_{r,R} := \inf_{\substack{t \in [t_0, t_1], x \in D \\ mr_1 \leq u \leq R_1, mr_2 \leq v \leq R_2}} \frac{g(t, x, u, v)}{r_2}$$

and

$$(3.20) \quad f^R := \sup_{\substack{t \in [0, t_{\max}], x \in \Omega \\ 0 \leq u \leq R_1, 0 \leq v \leq R_2}} \frac{f(t, x, u, v)}{R_1}, \quad g^R := \sup_{\substack{t \in [0, t_{\max}], x \in \Omega \\ 0 \leq u \leq R_1, 0 \leq v \leq R_2}} \frac{g(t, x, u, v)}{R_2}.$$

Evidently, $f_{r,R}^0 \leq f_{r,R} \leq f^R$ and $g_{r,R}^0 \leq g_{r,R} \leq g^R$.

Lemma 3.16. *Let α, β be as in Example 3.10. Assume that the inequalities (3.16) are satisfied.*

(a) *If $|u| \leq R_1, |v| \leq R_2$, then*

$$\frac{|N_1(u, v)|}{R_1} \leq q_1 Q_1 C_1 + f^R(q_1 Q_1 C_2 + C_1), \quad \frac{|N_2(u, v)|}{R_2} \leq q_2 Q_2 C_1 + g^R(q_2 Q_2 C_2 + C_1),$$

where the constants

$$(3.21) \quad C_1 = \left| \int_0^{t_{\max}} S_2(\tau) \chi_\Omega d\tau \right|, \quad C_2 = \left| \int_0^{t_{\max}} \int_0^t S_2(\tau) \chi_\Omega d\tau dt \right|$$

are positive.

(b) *The following implications hold:*

$$\begin{aligned} \lfloor u \rfloor \geq r_1 &\implies \frac{\lfloor N_1(u, v) \rfloor}{r_1} \geq p_1 P_1 (c_1 + f_{r,R}^0 c_2), \\ \lfloor v \rfloor \geq r_2 &\implies \frac{\lfloor N_2(u, v) \rfloor}{r_2} \geq p_2 P_2 (c_1 + g_{r,R}^0 c_2), \end{aligned}$$

where the constants

$$(3.22) \quad c_1 = \left\lfloor \int_0^{t_{\max}} S_2(\tau) \chi_D d\tau \right\rfloor, \quad c_2 = \left\lfloor \int_{t_0}^{t_{\max}} \int_{t_0}^{\min\{t, t_1\}} S_2(t - \tau) \chi_D d\tau dt \right\rfloor$$

are positive.

(c) If $\lfloor u \rfloor \geq r_1$ and $\lfloor v \rfloor \geq r_2$, then

$$\frac{\lfloor N_1(u, v) \rfloor}{r_1} \geq p_1 P_1(c_1 + f_{r,R} c_2), \quad \frac{\lfloor N_2(u, v) \rfloor}{r_2} \geq p_2 P_2(c_1 + g_{r,R} c_2),$$

Proof. Consider u, v such that $|u| \leq R_1$, $|v| \leq R_2$.

(a) Using the symbols u_0, \bar{u}, \bar{v} introduced in the proof of Proposition 3.7 and having in mind the equality $u_0 = \alpha(\bar{u}, \bar{v})$ and the estimates (3.17) we obtain

$$(3.23) \quad \frac{1}{q_1 Q_1} u_0 \leq \int_0^{t_{\max}} \bar{u}(t) dt.$$

Exploiting the equation (3.10) we obtain

$$(3.24) \quad \int_0^{t_{\max}} \bar{u}(t) dt = \int_0^{t_{\max}} S(t) u(0) dt + \int_0^{t_{\max}} \int_0^t S(t - \tau) F(u, v)(\tau) d\tau dt.$$

From (3.23), (3.24), (3.20) and the fact that $u \leq R_1 \chi_\Omega$ it can be concluded that

$$(3.25) \quad \frac{1}{q_1 Q_1} u_0 \leq R_1 \int_0^{t_{\max}} S_2(t) \chi_\Omega dt + R_1 f^R \int_0^{t_{\max}} \int_0^t S_2(\tau) \chi_\Omega d\tau dt.$$

From (3.9), we obtain

$$(3.26) \quad N_1(u, v)(t) = S(t) u_0 + \int_0^t S(t - \tau) F(u, v)(\tau) d\tau \leq S(t) u_0 + R_1 f^R \int_0^{t_{\max}} S_2(\tau) \chi_\Omega d\tau.$$

Combining (3.25) and (3.26) and applying the contractiveness of $S(t)$ gives

$$\frac{|N_1(u, v)|}{R_1} \leq q_1 Q_1 C_1 + f^R (q_1 Q_1 C_2 + C_1).$$

Similarly, we can obtain the estimate of $|N_2(u, v)|$.

(b) Assume now that $\lfloor u \rfloor \geq r_1$. Then $u(0) \geq r_1 \chi_D$. From (3.15) and (3.18) we obtain

$$u(t) \geq m r_1 \chi_D \text{ and } F(u, v)(t) \geq r_1 f_{r,R}^0 \chi_D \text{ for } t \in [t_0, t_1].$$

Using the symbols \bar{u}, \bar{v} introduced in the proof of Proposition 3.7 we obtain

$$\begin{aligned} \frac{1}{p_1 P_1} N_1(u, v)(0) &= \frac{1}{p_1 P_1} \alpha(\bar{u}, \bar{v}) \geq \int_0^{t_{\max}} \bar{u}(t) dt \\ &= \int_0^{t_{\max}} S(t) u(0) dt + \int_0^{t_{\max}} \int_0^t S(t - \tau) F(u, v)(\tau) d\tau dt \geq \\ &\geq r_1 \int_0^{t_{\max}} S_2(t) \chi_D dt + r_1 f_{r,R}^0 \int_{t_0}^{t_{\max}} \int_{t_0}^{\min\{t, t_1\}} S_2(t - \tau) \chi_D d\tau dt. \end{aligned}$$

Using the superadditivity of $\lfloor \cdot \rfloor$ we obtain

$$\frac{\lfloor N_1(u, v) \rfloor}{r_1} \geq p_1 P_1(c_1 + f_{r,R}^0 c_2).$$

In the same manner we can obtain the estimate of $\lfloor N_2(u, v) \rfloor$.

(c) Using the fact, that $\lfloor v \rfloor \geq r_2$ implies $v(t) \geq mr_2\chi_D$ and following the calculations analogous to those above, we obtain the conclusion.

The positiveness of the constants c_1, c_2, C_1, C_2 follows from Corollary 3.14. \square

Remark 3.17. As it was pointed out in Remark 3.15, the Harnack-type inequality (3.15) is an analogue of the inequality (3.4) of [16]. These two inequalities play a crucial role in obtaining the estimates from below and are utilized for the calculation of the fixed point index on some suitable subsets of a cone.

The difference between these two Harnack-type inequalities deserves a comment, as our choice to use the inequality (3.15) led us to build the new theory presented in Section 2. The inequality (3.4) of [16] was directly derived from a Harnack-type inequality given by Trudinger [25]. The natural counterpart in our context would be the *parabolic* Harnack inequality by Trudinger [26, Theorem 1.2], which is valid for all weak supersolutions u of the equation $u_t - \Delta u = 0$ (that is, functions $u \geq 0$ such that $u_t - \Delta u \geq 0$). This inequality could be expressed in the following manner:

$$(3.27) \quad u \geq c \|u\| \chi_{Q_-},$$

where $c > 0$ is a constant and

$$\|u\| = \int_{Q_+} |u(t, x)| dt dx, \quad Q_+ = [t^0, t^1] \times D, \quad Q_- = [t_0, t_1] \times D$$

for $0 < t^0 < t^1 < t_0 < t_1 \leq t_{\max}$. However, the use of this inequality is somewhat unnatural and it seems that it leads to additional complication of the argument and to worse results. For the sake of brevity, we provide here only a brief explanation.

1) The inequality (3.27) gives a lower bound on the values of u on Q_- , which depend on the values of u on Q_+ . The proof of Lemma 3.16(b) shows that it is more convenient to utilize the dependence of values $u|_{Q_-}$ on values of $u|_{\{0\} \times D}$, due to the presence of the nonlocal boundary condition $u(0, x) = \alpha(u, v)(x)$.

2) The inequality (3.15) is actually a consequence of the inequality $u \geq m \lfloor u \rfloor \chi_{[t_0, t_1] \times D}$ used for $u(t) = S_2(t)\chi_D$, which follows from the very definition of m . On the other hand, the inequality (3.27) is more general in the sense that the constant c is so chosen that a supersolution u of the equation $u_t - \Delta u = 0$ satisfies the estimate $u(t, x) \geq c$ on Q_- whenever $\|u\| \geq 1$. This universality, which in other context proves to be very important, is not exploited in our consideration, and unfortunately, it effects in a negative way the constants arising in the lower bounds of the nonlinearities (the counterparts of c_1 and c_2). And lastly, those constants are more difficult to be established. In other words, having in mind the nature of the calculations from Lemma 3.16, the use of minimum $\lfloor u \rfloor$ is more convenient, natural and effective than the use of the integral seminorm $\|u\|$.

3.1. Existence results. We are now prepared to establish some sufficient conditions for the existence of nonnegative nontrivial solutions of the problem (3.1). In what follows we shall assume that α, β are as in Example 3.10 and that the estimates (3.16) is satisfied.

Theorem 3.18. Assume there exist radii $0 < r_i < R_i$, $i \in \{1, 2\}$ such that

$$(3.28) \quad q_1 Q_1 C_1 + f^R(q_1 Q_1 C_2 + C_1) \leq 1, \quad q_2 Q_2 C_1 + g^R(q_2 Q_2 C_2 + C_1) \leq 1$$

and

$$(3.29) \quad p_1 P_1(c_1 + f_{r, R} c_2) > 1, \quad p_2 P_2(c_1 + g_{r, R} c_2) > 1.$$

Then there exists at least one nonnegative solution (u, v) of the problem (3.1) such that $|u| \leq R_1$, $|v| \leq R_2$ and $\lfloor u \rfloor > r_1$, $\lfloor v \rfloor > r_2$.

Proof. Lemma 3.16 implies that the conditions (2.4) and (2.5) are satisfied. Moreover, $r_i < \lfloor \psi_i \rfloor R_i$, since $\lfloor \psi_i \rfloor = 1$. The assertion follows from Theorem 2.6. \square

Remark 3.19. It is worth mentioning that the condition $\lfloor u \rfloor > r_1$ implies $|u| > r_1$, which follows from both the definitions of $|\cdot|$ and $\lfloor \cdot \rfloor$ and from Remark 2.4.

Theorem 3.20. Let $0 < r_1 < R_1$, $0 < r_2 < R_2$ and let $\tilde{R}_1 \leq R_1$, $\tilde{R}_2 \leq R_2$. Define the quantities

$$f_{r, \tilde{R}}^{00} := \inf_{\substack{t \in [t_0, t_1], \ x \in D \\ 0 \leq u \leq \tilde{R}_1, \ 0 \leq v \leq \tilde{R}_2}} \frac{f(t, x, u, v)}{r_1}, \quad g_{r, \tilde{R}}^{00} := \inf_{\substack{t \in [t_0, t_1], \ x \in D \\ 0 \leq u \leq \tilde{R}_1, \ 0 \leq v \leq \tilde{R}_2}} \frac{g(t, x, u, v)}{r_2}.$$

Assume that the condition (3.28) is satisfied and that

$$(3.30) \quad f_{r, \tilde{R}}^{00} \geq (p_1 P_1 c_2)^{-1} \quad \text{or} \quad g_{r, \tilde{R}}^{00} \geq (p_2 P_2 c_2)^{-1}.$$

Then there exists a nontrivial nonnegative solution (u, v) of (3.1) such that $|u| \leq R_1$, $|v| \leq R_2$ and

$$(3.31) \quad \lfloor u \rfloor \geq r_1 \quad \text{or} \quad \lfloor v \rfloor \geq r_2 \quad \text{or} \quad |u| > \tilde{R}_1 \quad \text{or} \quad |v| > \tilde{R}_2.$$

In particular, if $\tilde{R}_1 = R_1$ and $\tilde{R}_2 = R_2$, then there exists a nontrivial nonnegative solution (u, v) with either $\lfloor u \rfloor \geq r_1$ or $\lfloor v \rfloor \geq r_2$.

Proof. As in the previous proof we know that the condition (2.12) from Theorem 2.7 is satisfied. Let

$$(u, v) \in A := \left\{ (u, v) : \lfloor u \rfloor < r_1, \ \lfloor v \rfloor < r_2, \ |u| \leq \tilde{R}_1, \ |v| \leq \tilde{R}_2 \right\}.$$

In the same way as in the proof of Lemma 3.16(b) we can demonstrate that

$$(3.32) \quad \frac{\lfloor N_1(u, v) \rfloor}{r_1} \geq p_1 P_1 c_2 f_{r, \tilde{R}}^{00}, \quad \frac{\lfloor N_2(u, v) \rfloor}{r_2} \geq p_2 P_2 c_2 g_{r, \tilde{R}}^{00} \quad \text{for } |u| \leq \tilde{R}_1 \text{ and } |v| \leq \tilde{R}_2.$$

Therefore, the assumption (3.30) implies the condition (2.13) is satisfied and we can apply Theorem 2.7 to obtain a solution $(u, v) \notin A$. Clearly, this is equivalent to (3.31). \square

Remark 3.21. The importance of Theorem 3.20 consists in the fact that the assumption (3.30) involves only one component of the system nonlinearity (f, g) . Therefore, it allows different kind of growth of f and g near the origin. A similar remark also applies to the following theorem.

Theorem 3.22. Assume there exist radii $0 < \rho_i < r_i < R_i$, $i \in \{1, 2\}$, such that

$$q_1 Q_1 C_1 + f^R(q_1 Q_1 C_2 + C_1) \leq 1, \quad q_2 Q_2 C_1 + g^R(q_2 Q_2 C_2 + C_1) \leq 1,$$

$$q_1 Q_1 C_1 + f^\rho(q_1 Q_1 C_2 + C_1) \leq 1, \quad q_2 Q_2 C_1 + g^\rho(q_2 Q_2 C_2 + C_1) \leq 1$$

and

$$(3.33) \quad p_1 P_1(c_1 + f_{r,R} c_2) > 1, \quad p_2 P_2(c_1 + g_{r,R} c_2) > 1$$

Then there exist three nonnegative solutions (u_i, v_i) ($i = 1, 2, 3$) of the system (3.1) with

$$\begin{aligned} &|u_1| < \rho_1, \quad |v_1| < \rho_2 \text{ (possibly the zero solution);} \\ &\lfloor u_2 \rfloor < r_1 \text{ or } \lfloor v_2 \rfloor < r_2; \quad |u_2| > \rho_1 \text{ or } |v_2| > \rho_2 \text{ (possibly one solution component zero);} \\ &\lfloor u_3 \rfloor > r_1, \quad \lfloor v_3 \rfloor > r_2 \text{ (both solution components nonzero).} \end{aligned}$$

By the following slight strengthening of the assumption (3.33):

$$(3.34) \quad p_1 P_1(c_1 + f_{r,R}^0 c_2) > 1, \quad p_2 P_2(c_1 + g_{r,R}^0 c_2) > 1$$

we obtain a slight improvement of the precision in localizing the second solution:

$$\lfloor u_2 \rfloor < r_1, \quad \lfloor v_2 \rfloor < r_2; \quad |u_2| > \rho_1 \text{ or } |v_2| > \rho_2.$$

Moreover, having given numbers $0 < \varrho_i < \rho_i$ ($i = 1, 2$),

(i) if

$$p_1 P_1(c_1 + f_{\varrho,\rho} c_2) > 1, \quad p_2 P_2(c_1 + g_{\varrho,\rho} c_2) > 1,$$

then $\lfloor u_1 \rfloor \geq \varrho_1$ and $\lfloor v_1 \rfloor \geq \varrho_2$;

(ii) if

$$f_{\varrho,\tilde{\rho}}^{00} \geq (p_1 P_1 c_2)^{-1} \quad \text{or} \quad g_{\varrho,\tilde{\rho}}^{00} \geq (p_2 P_2 c_2)^{-1}$$

for some $\tilde{\rho}_1 \leq \rho_1$, $\tilde{\rho}_2 \leq \rho_2$, then $\lfloor u_1 \rfloor \geq \varrho_1$ or $\lfloor v_1 \rfloor \geq \varrho_2$ or $|u_1| > \tilde{\rho}_1$ or $|v_1| > \tilde{\rho}_2$.

Proof. One can use Lemma 3.16. The first assertion follows from Theorem 2.8, while second follows from Theorem 2.9. The third part of the conclusion, i.e. assertions (i) and (ii), is a consequence of Theorem 2.10 and (3.32). \square

3.2. Non-existence results. We now present some sufficient conditions for the non-existence of positive solutions of the system (3.1). We still assume that α, β are as in Example 3.10 and that (3.16) is satisfied.

Lemma 3.23. *Let (u, v) be a solution of the system (3.1). If $u \neq 0$ then $\lfloor u \rfloor > 0$ and if $v \neq 0$ then $\lfloor v \rfloor > 0$.*

Proof. Let $0 \leq t_0 < t_{\max}$ be such that $u(t_0) \neq 0$. From Proposition 3.7 we know that (u, v) is a fixed point of M . Thus,

$$\lfloor u(0) \rfloor = \lfloor \alpha(u, v) \rfloor \geq p_1 P_1 \left[\int_0^{t_{\max}} u(t) dt \right] \geq p_1 P_1 \left[\int_{t_0}^{t_{\max}} u(t) dt \right]$$

and $u(t) \geq S(t - t_0)u(t_0)$ for $t \geq t_0$. From Lemma 3.13 we therefore obtain

$$\lfloor u \rfloor = \lfloor u(0) \rfloor \geq p_1 P_1 \left[\int_0^{t_{\max}-t_0} S(t)u(t_0) dt \right] > 0.$$

\square

Theorem 3.24. *Put*

$$\bar{e}_i = \max \left(\frac{1 - q_i Q_i C_1}{q_i Q_i C_2 + C_1}, \frac{1 - q_i Q_i t_{\max}}{C_1} \right), \quad \underline{e}_i = \frac{(p_i P_i)^{-1} - c_1}{m c_2}, \quad i \in \{0, 1\}.$$

Assume that (u_0, v_0) is a nonnegative solution of the system (3.1). If one of the following conditions holds:

$$(3.35) \quad f(t, x, u, v) < \bar{e}_1 u \text{ for all } t \in [0, t_{\max}], \quad x \in \Omega, \quad u > 0, \quad v \geq 0,$$

$$(3.36) \quad f(t, x, u, v) > \underline{e}_1 u \text{ for all } t \in [t_0, t_1], \quad x \in D, \quad u > 0, \quad v \geq 0,$$

then $u_0 = 0$.

Similarly, if one of the following conditions holds:

$$(3.37) \quad g(t, x, u, v) < \bar{e}_2 v \text{ for all } t \in [0, t_{\max}], \quad x \in \Omega, \quad u \geq 0, \quad v > 0,$$

$$(3.38) \quad g(t, x, u, v) > \underline{e}_2 v \text{ for all } t \in [t_0, t_1], \quad x \in D, \quad u \geq 0, \quad v > 0,$$

then $v_0 = 0$.

In particular, if $p_1 P_1 c_1 > 1$ ($p_2 P_2 c_1 > 1$), then $u = 0$ ($v = 0$), regardless of the properties of f (g).

Proof. Suppose on the contrary that $u \neq 0$. Then $u_0 = N_1(u_0, v_0) = M_1(u_0, v_0)$. Put $R_1 = |u_0|$, $R_2 = \max\{|v_0|, 1\}$ and $r_1 = \lfloor u_0 \rfloor > 0$.

Assume that the inequality (3.35) holds. Observe that $f(t, x, u, v)/R_1 < \bar{e}_1$ for $t \in [0, t_{\max}]$, $x \in \bar{\Omega}$, $0 \leq u \leq R_1$ and $0 \leq v \leq R_2$. Therefore, $f^R < \bar{e}_1$. From Lemma 3.16(a) we obtain

$$(3.39) \quad 1 = \frac{|u_0|}{R_1} = \frac{|N_1(u_0, v_0)|}{R_1} \leq q_1 Q_1 C_1 + f^R (q_1 Q_1 C_2 + C_1).$$

Similarly, in the analogous manner as in the proof of Lemma 3.16(a), one can show that

$$(3.40) \quad 1 = \frac{|u_0|}{R_1} = \frac{|M_1(u_0, v_0)|}{R_1} \leq q_1 Q_1 t_{\max} + f^R C_1.$$

The estimates (3.39) and (3.40) give $f^R \geq \bar{e}_1$, a contradiction.

Assume now that (3.36) holds. Observe that $f(t, x, u, v)/r_1 > m \underline{e}_1$ for $t \in [t_0, t_1]$, $x \in D$, $m r_1 \leq u \leq R_1$ and $0 \leq v \leq R_2$. Therefore, $f_{r,R}^0 > m \underline{e}_1$.

On the other hand, from Lemma 3.16(b) we obtain

$$(3.41) \quad 1 = \frac{|u_0|}{r_1} = \frac{|N_1(u_0, v_0)|}{r_1} \geq p_1 P_1 (c_1 + f_{r,R}^0 c_2).$$

From (3.41) we conclude that $f_{r,R}^0 \leq m \underline{e}_1$, a contradiction.

The second assertion can be proved analogously. □

Corollary 3.25. (i) *If one of the inequalities (3.35)-(3.38) holds, then there are no positive solutions of the system (3.1).*

(ii) *If one of the inequalities (3.35)-(3.36) holds and one of the inequalities (3.37)-(3.38) holds, then there are no nontrivial nonnegative solutions of the system (3.1).*

3.3. An example. In this one-dimensional example we show that all the constants C_1, C_2, c_1, c_2, d that occur in our theory can be computed.

Let $\Omega = [0, \pi]$. Let us put $D = [b, \pi - b]$ for a fixed $0 < b < \pi/2$. Then

$$\chi_D = \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi} \cos((2k+1)b) \cdot \sin((2k+1)x)$$

and

$$\chi_{\Omega} = \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi} \sin((2k+1)x),$$

where the convergence is considered in the space $L^2(\Omega)$.

The evolution of χ_D and χ_{Ω} is given by the formulae

$$(S_2(t)\chi_D)(x) = \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi} \cos((2k+1)b) \cdot e^{-(2k+1)^2 t} \cdot \sin((2k+1)x),$$

$$(S_2(t)\chi_{\Omega})(x) = \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi} \cdot e^{-(2k+1)^2 t} \cdot \sin((2k+1)x).$$

Put $b = \pi/4$, $t_0 = 0$ and $t_1 = t_{\max} = 1$. Then we have

$$m = (S_2(1)\chi_D)\left(\frac{\pi}{4}\right) = \sum_{k=0}^{\infty} (-1)^k \frac{2}{(2k+1)\pi} e^{-(2k+1)^2} \approx 0.23.$$

Because

$$\int_0^{t_{\max}} S_2(\tau)\chi_D d\tau(x) = \sum_{k=0}^{\infty} \frac{4}{(2k+1)^3\pi} \cos(2k+1)\frac{\pi}{4} \cdot \left(1 - e^{-(2k+1)^2}\right) \cdot \sin((2k+1)x)$$

and

$$\int_0^{t_{\max}} \int_0^t S_2(t-\tau)\chi_D d\tau dt = \sum_{k=0}^{\infty} \frac{4}{(2k+1)^5\pi} \cos(2k+1)\frac{\pi}{4} \cdot e^{-(2k+1)^2} \cdot \sin((2k+1)x),$$

we obtain

$$c_1 = \sum_{k=0}^{\infty} \frac{2 \cdot (-1)^k}{(2k+1)^3\pi} \left(1 - e^{-(2k+1)^2}\right) \approx 0.38$$

and

$$c_2 = \sum_{k=0}^{\infty} \frac{2 \cdot (-1)^k}{(2k+1)^5\pi} e^{-(2k+1)^2} \approx 0.23$$

Moreover, because

$$\int_0^{t_{\max}} S_2(\tau)\chi_{\Omega} d\tau = \sum_{k=0}^{\infty} \frac{4}{(2k+1)^3\pi} \cdot \left(1 - e^{-(2k+1)^2}\right) \cdot \sin((2k+1)x)$$

and

$$\int_0^{t_{\max}} \int_0^t S_2(\tau)\chi_{\Omega} d\tau dt = \sum_{k=0}^{\infty} \frac{4}{(2k+1)^5\pi} \cdot e^{-(2k+1)^2} \cdot \sin((2k+1)x),$$

$$(3.42) \quad C_1 = \left| \int_0^{t_{\max}} S_2(\tau)\chi_{\Omega} d\tau \right|, \quad C_2 = \left| \int_0^{t_{\max}} \int_0^t S_2(\tau)\chi_{\Omega} d\tau dt \right|$$

we obtain

$$C_1 = \sum_{k=0}^{\infty} \frac{4 \cdot (-1)^k}{(2k+1)^3 \pi} \cdot \left(1 - e^{-(2k+1)^2}\right) \approx 0.77$$

and

$$C_2 = \sum_{k=0}^{\infty} \frac{4 \cdot (-1)^k}{(2k+1)^5 \pi} \cdot e^{-(2k+1)^2} \approx 0.47.$$

If we put $G_i(x) = g_i(x) = x$, then $p_i = P_i = q_i = Q_i = 1$ and the conditions (3.28) and (3.29) are equivalent to the following inequalities:

$$f^R, g^R \leq \frac{1 - C_1}{C_1 + C_2} \approx 0.19, \quad f_{r,R}, g_{r,R} > \frac{1 - c_1}{c_2} \approx 2.64.$$

The numerical calculations indicate that the choice $b = \pi/4$ is optimal, i.e. the ratio $(1 - c_1)/(c_2 m)$ is the smallest.

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REFERENCES

- [1] W. Arendt, C. Batty, M. Hieber and F. Neubrander, *Vector-valued Laplace transforms and Cauchy problems*. Monographs in Mathematics, 96. Birkhäuser Verlag, Basel, 2001.
- [2] I. Benedetti, V. Taddei and M. Văth, Evolution problems with nonlinear nonlocal boundary conditions, *J. Dynam. Differential Equations* **25** (2013), 477–503.
- [3] A. Boucherif, Nonlocal problems for parabolic inclusions, *Discrete Contin. Dyn. Syst.*, **suppl.** (2009), 82–91.
- [4] A. Boucherif, Discontinuous parabolic problems with a nonlocal initial condition, *Bound. Value Probl.*, **2011**, Art. ID 965759, 10 pp.
- [5] A. Boucherif, Nonlocal conditions for lower semicontinuous parabolic inclusions, *Adv. Difference Equ.*, **2011**, Art. ID 109570, 7 pp.
- [6] T. Cardinali, R. Precup and P. Rubbioni, A unified existence theory for evolution equations and systems under nonlocal conditions, *arXiv:1406.6825 [math.CA]*, 2014.
- [7] J. Chabrowski, On nonlocal problems for parabolic equations, *Nagoya Math. J.*, **93** (1984), 109–131.
- [8] R. Dautray and J. L. Lions, *Mathematical analysis and numerical methods for science and technology*, Vol. 1–3. Springer-Verlag, Berlin, 1990.
- [9] K. Deimling, *Nonlinear functional analysis*, Springer-Verlag, Berlin, 1985.
- [10] K. Deng, Exponential decay of solutions of semilinear parabolic equations with nonlocal initial conditions, *J. Math. Anal. Appl.*, **179** (1993) 630–637.
- [11] K. J. Engel and R. Nagel, *One-parameter semigroups for linear evolution equations*. Graduate Texts in Mathematics, 194. Springer-Verlag, New York, 2000.

- [12] L. C. Evans, *Partial differential equations*. Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 1998.
- [13] A. Granas and J. Dugundji, *Fixed point theory*, Springer-Verlag, New York, 2003.
- [14] D. Guo and V. Lakshmikantham, *Nonlinear problems in abstract cones*, Academic Press, Boston, 1988.
- [15] G. Infante, Nonlocal boundary value problems with two nonlinear boundary conditions, *Commun. Appl. Anal.*, **12** (2008), 279-288.
- [16] G. Infante, M. Maciejewski and R. Precup, A topological approach to the existence and multiplicity of positive solutions of (p, q) -Laplacian systems, *arXiv:1401.1355 [math.AP]*, 2014.
- [17] G. Infante and P. Pietramala, Existence and multiplicity of non-negative solutions for systems of perturbed Hammerstein integral equations, *Nonlinear Anal.*, **71** (2009), 1301–1310.
- [18] M. A. McKibben, *Discovering evolution equations with applications*, vol. I. Chapman & Hall Mathematics, Boca Raton, 2011.
- [19] W. E. Olmstead and C. A. Roberts, The one-dimensional heat equation with a nonlocal initial condition, *Appl. Math. Lett.*, **10** (1997), 89–94.
- [20] C. V. Pao, Reaction diffusion equations with nonlocal boundary and nonlocal initial conditions, *J. Math. Anal. Appl.*, **195** (1995), 702–718.
- [21] R. Precup, Parabolic Harnack type inequalities and multiple positive solutions of evolution equations (preprint).
- [22] J. M. Rassias and E. T. Karimov, Boundary-value problems with non-local initial condition for parabolic equations with parameter, *European J. Pure Appl. Math.*, **3** (2010) 948–957.
- [23] J. M. Rassias and E. T. Karimov, Boundary-value problems with non-local condition for degenerate parabolic equations, *Contemporary Analysis and Applied Mathematics*, Vol.1, No.1, 42-48, 2013.
- [24] A. Štikonas, A survey on stationary problems, Green's functions and spectrum of Sturm-Liouville problem with nonlocal boundary conditions, *Nonlinear Anal. Model. Control*, **19** (2014), 301–334.
- [25] N. S. Trudinger, On Harnack type inequalities and their applications to quasilinear elliptic equations, *Comm. Pure Appl. Math.*, **20** (1967), 721–747.
- [26] N. S. Trudinger, Pointwise estimates and quasilinear parabolic equations, *Comm. Pure Appl. Math.*, **21** (1968), 205-226.
- [27] I. I. Vrabie, *C_0 -semigroups and applications*. North-Holland Mathematics Studies, 191. North-Holland Publishing Co., Amsterdam, 2003.

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